Proper $\mathscr{L}(h,k)$-labelling of Caterpillars and Multisunlets[1]

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Abstract

An $\mathscr{L}(h,k)$-*labelling* of a simple graph G is a function $\sigma\colon V(G) \to \mathbb{Z}_{\geq 0}$ such that the labels of: adjacent vertices are at least h apart; vertices which have a common neighbour are at least k apart. The *span* of σ is the largest difference between the labels of any two vertices. Computing the $\mathscr{L}(h,k)$-span of G, the least span amongst all σ, is NP-hard even for trees. For $h \geq k$, we determine the $\mathscr{L}(h,k)$-span of the *Uniform Multisunlets* (*Uniform Caterpillars*), in which $p \geq 1$ pendant vertices are added at each vertex of a basis cycle (path).

Keywords: Graph labelling, Caterpillar graphs, Sunlet graphs

1. Introduction

Let G be a simple graph with *vertex* set $V(G)$ and *edge* set $E(G)$. An edge $e \in E(G)$ with *endpoints* u and v is denoted uv. The *degree* of $u \in V(G)$ is denoted $d(u)$. A *pendant* is a vertex of degree one. The set of *neighbours* of u is denoted $N(u)$.

The $\mathscr{L}(2,1)$-labelling, proposed to approach the Frequency Assignment Problem [1], is an assignment of integers to the vertices of a graph wherein adjacent vertices receive labels that are at least two apart and non-adjacent vertices having a common neighbour receive different labels. Later, Georges and Mauro [2] generalised this concept as presented in Definition 1.

Definition 1. Let $h,k \in \mathbb{Z}_{\geq 0}$ and G be a simple graph. An $\mathscr{L}(h,k)$-*labelling* of G is a function $\sigma\colon V(G) \to \mathbb{Z}_{\geq 0}$ such that:

(i) $|\sigma(u) - \sigma(v)| \geq h$, for all $uv \in E(G)$;

(ii) $|\sigma(u) - \sigma(v)| \geq k$, for all distinct $u,v \in V(G)$ with $N(u) \cap N(v) \neq \emptyset$.

The *span* of an $\mathscr{L}(h,k)$-labelling σ is $\lambda(\sigma) = \max_{u,v \in V(G)}\{\sigma(u) - \sigma(v)\}$. The $\mathscr{L}(h,k)$-*span* of G is $\lambda_{h,k}(G) = \min_\sigma\{\lambda(\sigma)\}$. Some authors define (ii) in Definition 1 as $|\sigma(u) - \sigma(v)| \geq k$ *for all $u,v \in V(G)$ with $dist(u,v) = 2$*. Note that the two definitions are equivalent when

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$h \geq k$ or when G is triangle-free. On that account, we say that σ is a *proper* $\mathscr{L}(h,k)$-labelling if $h \geq k$.

The problem of determining the exact value of the $\mathscr{L}(h,k)$-span is completely settled only for basic graph classes, such as stars, paths, and cycles, or specific values of h and k [2, 3]. On the other hand, this problem is NP-hard even for trees whenever k does not divide h [4].

Besides paths and stars, another important subclass of trees are the *Caterpillars*. These are trees for which there is a path P such that every vertex either lies in P or is adjacent to a vertex in P. We approach the proper $\mathscr{L}(h,k)$-labelling problem in this class, determining the $\mathscr{L}(h,k)$-span of *Uniform Caterpillars* $\check{P}_{n,p}$, in which $p \geq 1$ pendant vertices are added at each vertex of a path of n vertices, called the *basis* of $\check{P}_{n,p}$, as in Figure 1.

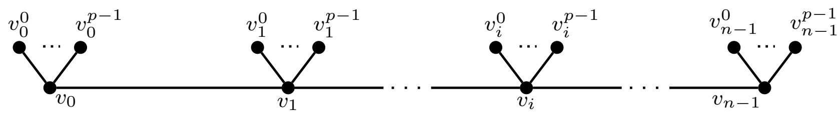

Figure 1: The Uniform Caterpillar $\check{P}_{n,p}$.

As Caterpillars generalise paths, an interesting generalisation of the cycles are the *Sunlets* $\acute{C}_n$, also much studied in labelling problems [5]. These are the graphs obtained by adding one pendant vertex at each vertex of a cycle of n vertices, called the *basis* of $\acute{C}_n$. We not only settle the proper $\mathscr{L}(h,k)$-labelling problem for the Sunlets, but we also determine the span of *Uniform Multisunlets* $\check{C}_{n,p}$, in which $p \geq 1$ pendant vertices are added at each vertex of a cycle of n vertices, the *basis* of $\check{C}_{n,p}$.

2. Preliminaries

In an $\mathscr{L}(h,k)$-labelling, the set of all neighbours of some $u \in V(G)$ whose labels are smaller (greater) than the label of u is denoted $N_<(u)$ ($N_>(u)$). For $n, p \in \mathbb{Z}_{\geq 0}$, the vertices of the basis of $\check{P}_{n,p}$ ($\check{C}_{n,p}$) are denoted v_i for $0 \leq i < n$, and the pendant vertices at each v_i are denoted v_i^j for $0 \leq j < p$, as in Figure 1. For $n \in \mathbb{Z}_{\geq 2}$, the *centre* of the *star* $K_{1,n}$ is the vertex with degree n. We use $[a \,..\, b]$ to denote the set $\{i \in \mathbb{Z} : a \leq i \leq b\}$.

Since an $\mathscr{L}(h,k)$-labelling of a graph G induces an $\mathscr{L}(h,k)$-labelling of every subgraph of G, and since each vertex of the basis of $\check{P}_{n,p}$ ($\check{C}_{n,p}$) and its neighbours induce a subgraph with a spanning star, we establish some properties of $\mathscr{L}(h,k)$-labellings of stars with special values of span.

Theorem 1 ([3]). *Let $h, k \in \mathbb{Z}_{\geq 0}$ and $p \in \mathbb{Z}_{>0}$ such that $h \geq k$. Then, $\lambda_{h,k}(K_{1,p}) = h + (p-1)k$.* □

Lemma 1. *Let $\alpha, \beta, i, h, k, p \in \mathbb{Z}_{\geq 0}$ with: either $\alpha = 1$, $i \geq 2$, $h \geq ik$, and $\beta \leq p+i$; or $\alpha = 2$ and $\beta \leq p$. If there is an $\mathscr{L}(h,k)$-labelling σ of $K_{1,p+2}$ with span $\alpha h + \beta k - 1$, then, being u the centre of $K_{1,p+2}$, one of the following holds: $N_<(u) = \emptyset$ and $\sigma(u) < (\alpha-1)h + (\beta-p-1)k$; $N_>(u) = \emptyset$ and $\sigma(u) \geq h + (p+1)k$.* □

Lemma 2. *Let $h, k, a, \gamma, p \in \mathbb{Z}_{\geq 0}$ with $0 < k < h < 2k$, $a = \max\{0, \lceil 3k/2 \rceil - h\}$, and: $\gamma = 1$ if $3k$ is odd and $h < 3k/2$; $\gamma = 0$ if otherwise. Let $\mathcal{X}_i = [ik - 2a + \gamma \mathinner{..} h + (i-1)k - 1]$ for $i \in [1 \mathinner{..} p+2]$. If there is an $\mathscr{L}(h,k)$-labelling of $K_{1,p+2}$ with span $\min\{h + (p+2)k - 1, 3h + (p-1)k - 1\}$, then the label of the centre of $K_{1,p+2}$ does not belong to $\mathcal{X}_i$ for any i.* □

Lemma 3. *Let $h, k, a, \gamma, p \in \mathbb{Z}_{\geq 0}$ with $0 < k < h < 2k$, $a = \max(0, \lceil 3k/2 \rceil - h)$, and: $\gamma = 1$ if $3k$ is odd and $h < 3k/2$; $\gamma = 0$ if otherwise. Let σ be an $\mathscr{L}(h,k)$-labelling of $K_{1,p+2}$ such that $\lambda(\sigma) = \min\{h + (p+2)k - 1, 3h + (p-1)k - 1\}$. Let $\mathcal{Y}_i = [h + (i-1)k \mathinner{..} (i+1)k - 2a + \gamma - 1]$ for $i \in [1 \mathinner{..} p+1]$. Let u be the centre of $K_{1,p+2}$. If $\sigma(u) \in \mathcal{Y}_i$ for some $i \in [1 \mathinner{..} p+1]$, then, for all $v \in V(K_{1,p+2}) \setminus \{u\}$, $\sigma(v) \notin \mathcal{Y}_j$ for any $j \in [1 \mathinner{..} p+1]$.* □

3. Main results

In Theorems 2 and 3, we establish $\lambda_{h,k}(\check{P}_{n,p})$ and $\lambda_{h,k}(\check{C}_{n,p})$, respectively.

Theorem 2. *Let $h, k, n, p \in \mathbb{Z}_{\geq 0}$ such that $h \geq k$, $p \geq 1$, and $n \geq 2$. Then, $\lambda_{h,k}(\check{P}_{n,p}) = \min\{h + (p+i)k, 2h + pk\}$, wherein $i = \min\{2, \lceil n/2 \rceil - 1\}$.*

Proof (sketch). If $2 \leq n \leq 4$, then $h + (p+i)k \leq 2h + pk$. In this case, the lower bound follows from Theorem 1, since $K_{1,p+i+1} \subseteq \check{P}_{n,p}$, and the upper bound follows by inspection on the labellings presented in Figure 2.

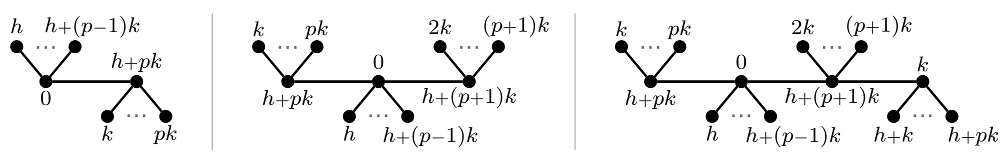

Figure 2: $\mathscr{L}(h,k)$-labelling of $\check{P}_{n,p}$ in the case wherein $2 \leq n \leq 4$

In order to show the lower bound for $n \geq 5$, assume that there is an $\mathscr{L}(h,k)$-labelling σ for $\check{P}_{n,p}$ with $\lambda(\sigma) = \min\{h + (p+2)k, 2h + pk\} - 1$. If $h + (p+2)k \leq 2h + pk$, then divide the available labels into three sets $\mathcal{X}_1 = [0 \mathinner{..} k-1]$, $\mathcal{X}_2 = [k \mathinner{..} h + (p+1)k - 1]$, and $\mathcal{X}_3 = [h + (p+1)k \mathinner{..} h + (p+2)k - 1]$. Since $\sigma(v_0)$, $\sigma(v_1)$, and $\sigma(v_3)$ are centres of a $K_{1,p+2}$, by Lemma 1, these vertices cannot be labelled with elements of $\mathcal{X}_2$. Also, we cannot have two of these three vertices simultaneously labelled with elements of either $\mathcal{X}_1$ or $\mathcal{X}_3$. Therefore, one of these vertices cannot be assigned any label. The reasoning for case $2h + pk \leq h + (p+2)k$ is similar.

For the upper bound, we present an $\mathscr{L}(h,k)$-labelling σ with same span as in the statement for: $\check{P}_{3t,p}$, with $t = \lceil n/3 \rceil$, if $\lambda(\sigma) = 2h + pk$; $\check{P}_{4t,p}$, with $t = \lceil n/4 \rceil$, if $\lambda(\sigma) = h + (p+2)k$. This labelling is constructed by replicating the blocks of Figure 3. Observing that $\check{P}_{n,p}$ is a subgraph of $\check{P}_{N,p}$ for any $N \geq n$, we conclude $\lambda_{h,k}(\check{P}_{n,p}) \leq \min\{h + (p+2)k, 2h + pk\}$. □

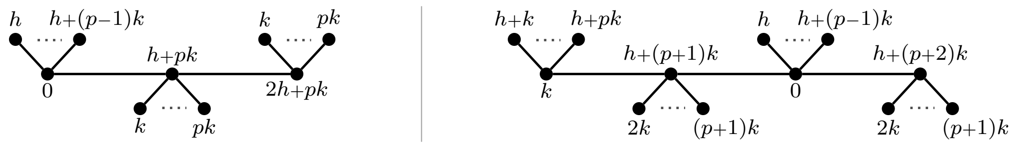

Figure 3: The blocks for our construction of an $\mathscr{L}(h,k)$-labelling with span: $2h+pk$ for $\check{P}_{3t,p}$ (left); $h+(p+2)k$ for $\check{P}_{4t,p}$ (right).

Theorem 3. *Let $h,k \in \mathbb{Z}_{\geq 0}$ such that:* (a) $0 \leq 2k \leq h$; *or* (b) $0 < k \leq h < 2k$. *Let $n,p \in \mathbb{Z}_{>0}$ such that $n \geq 3$. Let $i = (n \bmod 4)/2$. Then,*

$$\lambda_{h,k}(\check{C}_{n,p}) = \begin{cases} \min\{h+(p+2)k, 3h+(p-1)k+\lfloor 1/p \rfloor h\} & \textit{if } n=5 \textit{ and } \text{(b)}, \\ \min\{h+(p+2+i)k, 2h+pk\} & \textit{if } n \textit{ is even and } \text{(a)}, \\ 2h+pk & \textit{in any other case.} \end{cases}$$

Proof (sketch). Initially, we consider $n \neq 5$. First, we construct an $\mathscr{L}(h,k)$-labelling with span $2h+pk$. For $n \in \{3,4,8\}$, the labelling is presented in Table 1. For $6 \leq n \leq 7$ and $n \geq 9$, we start from the labelling for: $\check{C}_{3,p}$ if $n \equiv 0 \pmod 3$; $\check{C}_{4,p}$ if $n \equiv 1 \pmod 3$; or $\check{C}_{8,p}$ if $n \equiv 2 \pmod 3$. Then, the remaining vertices are labelled in blocks of three, following Figure 4.

Table 1. An $\mathscr{L}(h,k)$-labelling of $\check{C}_{n,p}$ for $n \in \{3,4,8\}$. Here, $0 < j < p$.

n mod 3	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_0^0	v_0^j	v_1^0	v_1^j
0	0	$2h+pk$	$h+pk$	–	–	–	–	–	h	$h+jk$	k	$(j+1)k$
1	0	h	$2h+pk$	$h+pk$	–	–	–	–	$2h+pk$	$h+jk$	$2h$	$2h+jk$
2	0	h	$2h+pk$	$h+pk$	0	h	$2h+pk$	$h+pk$	$2h+pk$	$h+jk$	$2h$	$2h+jk$

n mod 3	v_2^0	v_2^j	v_3^0	v_3^j	v_4^0	v_4^j	v_5^0	v_5^j	v_6^0	v_6^j	v_7^0	v_7^j
0	k	$(j+1)k$	–	–	–	–	–	–	–	–	–	–
1	0	$h+jk$	k	$(j+1)k$	–	–	–	–	–	–	–	–
2	0	$h+jk$	k	$(j+1)k$	$2h+pk$	$h+jk$	$2h$	$2h+jk$	0	$h+jk$	k	$(j+1)k$

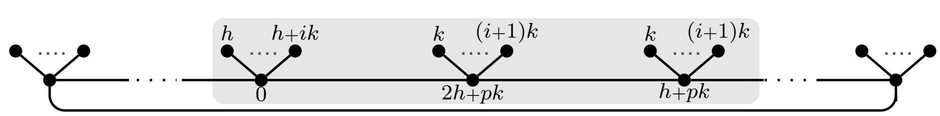

Figure 4: In grey, a building block for the labelling with span $2h+pk$ when $n \neq 5$.

For n even and $h \geq 2k$, we present another labelling, with span $h+(p+2+i)k$, wherein $i = (n \bmod 4)/2$. Note that $h+(p+2+i)k \leq 2h+pk$ if and only if $h \geq (2+i)k$. If $n \in \{4,6\}$, the labelling is presented in Table 2. If $n \geq 8$, we start from the labelling for: $\check{C}_{4,p}$ if $n \equiv 0$

Table 2. $\mathscr{L}(h,k)$-labelling of $\check{C}_{n,p}$ for $n \in \{4,6\}$. Here, $0 < j < p$.

n mod 4	v_0	v_1	v_2	v_3	v_4	v_5	v_0^0	v_0^j	v_1^0	v_1^j
0	0	$h+(p+2)k$	k	$h+(p+1)k$	–	–	h	$h+jk$	$2k$	$(j+2)k$
2	0	$h+(p+3)k$	k	$h+(p+1)k$	$2k$	$h+(p+2)k$	h	$h+jk$	$2k$	$(j+2)k$

n mod 4	v_2^0	v_2^j	v_3^0	v_3^j	v_4^0	v_4^j	v_5^0	v_5^j
0	$h+k$	$h+(j+1)k$	$2k$	$(j+2)k$	–	–	–	–
2	$h+k$	$h+(j+1)k$	0	$(j+2)k$	$h+(p+3)k$	$h+(j+1)k$	$3k$	$(j+3)k$

(mod 4); $\check{C}_{6,p}$ if $n \equiv 2 \pmod 4$. Then, for every $i \in [4+(n \bmod 4) \,..\, n-1]$, vertex v_i and its pendants are assigned the same labels of v_{i-4} and its pendants.

The proof of the lower bound is similar to that in Theorem 2. Being s the value established in the statement, we assume $\lambda(\check{C}_{n,p}) = s-1$ and partition the available labels into the three sets defined in Table 3, showing that at least one vertex of the basis cannot be assigned any label.

Table 3. Definition of sets $\mathcal{X}_1$, $\mathcal{X}_2$, and $\mathcal{X}_3$ when $n \neq 5$ or $h \geq 2k$ for cases: (i) n is even, $h \geq 2k$, and $i = (n \bmod 4)/2$; (ii) n in odd and $h \geq 2k$, or $n \neq 5$ and $h < 2k$.

	$\mathcal{X}_1$	$\mathcal{X}_2$	$\mathcal{X}_3$
(i)	$[0 \,..\, (1+i)k-1]$	$[(1+i)k \,..\, h+(p+1)k-1]$	$[h+(p+1)k \,..\, h+(p+2+i)k-1]$
(ii)	$[0 \,..\, h-k-1]$	$[h-k \,..\, h+(p+1)k-1]$	$[h+(p+1)k \,..\, 2h+pk-1]$

Consider, now, $n = 5$. The upper bound is established in Figure 5. The lower bound is quite challenging since the basis is a cycle with five vertices. By taking sets $\mathcal{X}_1, \ldots, \mathcal{X}_{p+2}$ and $\mathcal{Y}_1, \ldots, \mathcal{Y}_{p+1}$ as in Lemmas 2 and 3, we reach a contradiction using a similar technique to those presented so far. □

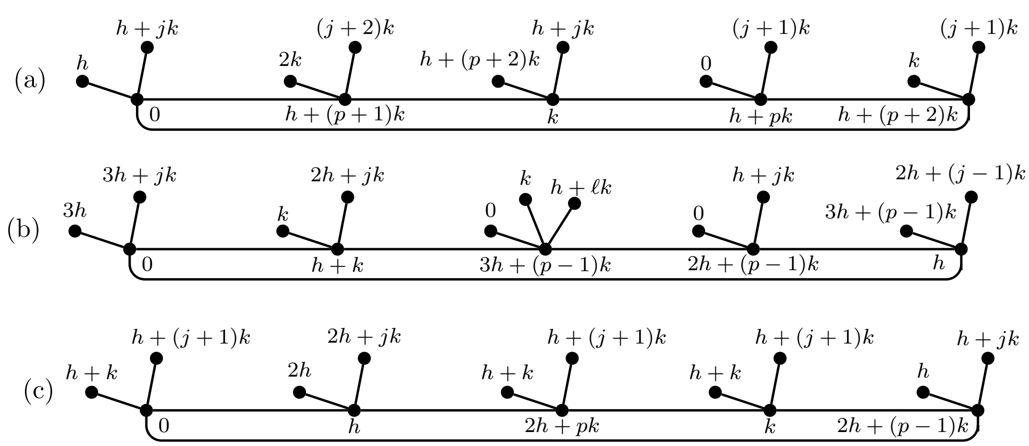

Figure 5: In (a)–(c), $0 < j < p$. In (a), $0 < k \leq h < 2k$ and: $p = 1$, or $p \geq 2$ and $h + (p+2)k \leq 3h + (p-1)k$; if $p = 1$, only the leftmost pendant exists. In (b), $0 < k \leq h < 2k$, $p \geq 2$, $3h + (p-1)k \leq h + (p+2)k$, and $1 < \ell < p$. In (c), $h \geq 2k$.

□

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