An Extended Adjacency Lemma for Graph Edge-Coloring

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Abstract

The study of edge-coloring emerged from the context of the well-known Four Color Problem. By Vizing's theorem, the minimum number of colors needed to color the edges of a simple graph G is either its maximum degree Δ, or it is $\Delta + 1$, in which case G is called Class 1, or Class 2. A critical graph is a connected Class 2 graph that becomes Class 1 by the removal of any edge. Vizing's recoloring procedure used in the proof of his theorem also yields a condition, known as Vizing's Adjacency Lemma, for the minimum number of vertices of degree Δ adjacent to any vertex in a critical graph. We demonstrate an extension of this adjacency lemma using an extended recoloring procedure presented recently.

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1. Introduction

Throughout this text, all graphs are simple. Let G be a graph, wherein $V(G)$ is the vertex set and $E(G)$ is the edge set of G. An edge uv in $E(G)$ is formed by two endpoints, $u, v \in V(G)$. The *degree* $d_G(u)$ of $u \in V(G)$ is the number of edges incident with u. The set of neighbors of u in G is denoted $N_G(u)$. Two edges are adjacent if they both are incident with some $u \in V(G)$. The maximum (minimum) degree of G is denoted $\Delta(G)$ ($\delta(G)$). Whenever free of ambiguity, we can omit the graph G from the notation, writing simply $d(u)$, Δ etc.

Vertices of maximum degree in G are referred to as the *majors* of G. The *core* of G, denoted $\Lambda[G]$, is the subgraph of G induced by its majors. The majors u of G with $\sum_{v \in N_G(u)} d_G(v) \geq \Delta^2 - \Delta + 2$ are the *proper* majors of G. If this sum is equal to $\Delta^2 - \Delta + 1$, then u is a *tightly non-proper* major of G. Otherwise, u is a *strictly non-proper* major of G. The *hard core* of G, denoted $\mathbf{\Lambda}[G]$, is the subgraph of G that is induced by all proper and tightly non-proper majors of G. Also, the *soft core* of graph G is the subgraph of G induced by the strictly non-proper majors of G.

An *edge-coloring* of a graph G is an assignment of colors to the edges of G so that no two adjacent edges are colored the same. For two colors α, β used in an edge-coloring, the *α/β-component* is the subgraph of G induced by the edges colored α or β. We say that a vertex u misses some color α if no edge incident with u in colored α.

The *chromatic index* of a graph G, denoted $\chi'(G)$, is the least amount of colors needed for an edge-coloring of G. The Edge-coloring Problem consists of finding the chromatic index of a given graph. This problem emerges from the context of the well-known Four Color Problem, which is equivalent to the statement that every planar cubic graph has chromatic index equal to three [1]. By Vizing's theorem [2], the chromatic index of a simple graph G is either $\Delta(G)$ or $\Delta(G)+1$, being G called Class 1 or Class 2, respectively. Deciding if a graph G is Class 1 is NP-complete [3].

The proof of Vizing's theorem follows from the recoloring procedure established in Definition 1 and Lemma 1. In both statements, G is a graph for which we already have an edge-coloring of $G-uv$ using t colors, for some $uv \in E(G)$ and some $t \in \mathbb{Z}_{>0}$.

Definition 1. A *Vizing's recoloring fan* for uv is a sequence $v_0, v_1, \ldots, v_k$ of distinct neighbors of u in G such that $v_0 = v$ and, for all $i \in \{0, \ldots, k-1\}$, vertex v_i misses the color α_i, which is the color of edge uv_{i+1}. This fan is said to be *complete* if both v_k and u miss the same color.

Lemma 1 ([2])**.** *If there is a Vizing's recoloring fan $v_0, v_1, \ldots, v_k$ for uv satisfying at least one of the properties below, then G also admits an edge-coloring using t colors.*

Properties 1a. *The fan $v_0, v_1, \ldots, v_k$ is complete.*

Properties 1b. *Vertex v_k misses the same color a_j as v_j in G for some $j < k$.*

With Vizing's recoloring procedure, we have another important result, known as Vizing's Adjacency Lemma. In the statement, a *critical* graph is a connected Class 2 graph G such that $\chi'(G-uv) < \chi'(G)$ for all $uv \in E(G)$.

Lemma 2 (Vizing's Adjacency Lemma [4])**.** *For every edge uv of a critical graph G, the number of majors adjacent to u in G is at least:*

$$\begin{aligned} \Delta - d_G(v) + 1, &\quad \textit{if } d_G(v) < \Delta; \\ 2, &\quad \textit{if } d_G(v) = \Delta. \end{aligned}$$

Recently, an extension of Vizing's recoloring procedure was proposed [5]. Definition 2 and Lemma 3 describe this procedure. In both statements, G is a graph for which we already have an edge-coloring of $G-uv$ using Δ colors, for some $uv \in E(G)$.

Definition 2. An *extended recoloring fan* for uv is a sequence $v_0, v_1, \ldots, v_k$ of distinct neighbors of u in G such that $v_0 = v$ and, for all $i \in \{0, \ldots, k-1\}$, either v_i actually misses color α_i, or $i > 0$ and v_i misses this color *virtually*, that is, there is an edge v_iw_i in G colored α_i such that $w_i \neq v_{i-1}$ actually misses α_{i-1}. This fan is said to be *complete* if v_k (actually or virtually) misses some color that is also missing at u (see Figure 1).

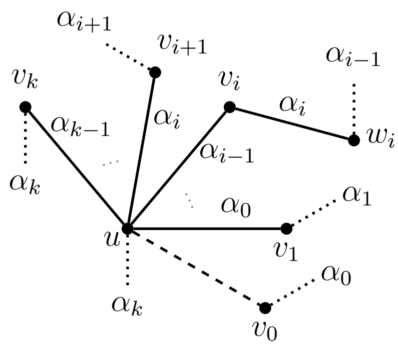

Figure 1: A complete extended recoloring fan

Lemma 3 ([5]). *If there is an extended recoloring fan $v_0, v_1, \ldots, v_k$ for uv satisfying at least one the properties below, then G is Class 1.*

Properties 3a. *The fan $v_0, v_1, \ldots, v_k$ is complete.*

Properties 3b. *Vertex v_k misses, virtually or actually, the same color a_j as v_j in G for some $j < k$.*

If there is an extended recoloring fan $v_0, v_1, \ldots, v_k$ that satisfies Property 3a, then the edge-coloring of G with Δ colors can be obtained by a procedure referred to as *decay of the colors* of the fan, which is performed for i from k down to 0. At the beginning of each iteration, it is invariant that both u and v_i miss α_i (the latter possibly virtually). So, uv_i is colored α_i and, if v_i misses α_i virtually, also $v_i w_i$ is colored α_{i-1}. If $i = 0$, we are done. If $i > 0$, now u misses α_{i-1}, which is still missing (possibly virtually) at v_{i-1}, so we can decrement i and continue.

Lemma 4 brings another interesting application of the extended recoloring procedure.

Lemma 4 ([5]). *Let G be a graph for which we already have an edge-coloring of $G - uv$ using Δ colors, for some $uv \in E(G)$. If all majors adjacent to u in $G - uv$ are strictly non-proper, then G is Class 1.*

In this paper, we show a stronger form of Lemma 2 with the use of the extended recoloring procedure.

2. Result

Lemma 5 (Extended Adjacency Lemma). *For every edge uv of a critical graph G, the number of vertices in $\mathbf{\Lambda}[G]$ adjacent to u is at least:*

$$\begin{aligned} \Delta - d_G(v) + 1, &\quad \text{if } d_G(v) < \Delta; \\ 2, &\quad \text{if } d_G(v) = \Delta. \end{aligned}$$

Proof. Consider the following:

Claim. Let uv be an edge of a graph G such that $\chi'(G - uv) = \Delta(G - uv) = \Delta(G) =: \Delta$. If u is adjacent in $G - uv$ to at most $\Delta - d_G(v)$ vertices in $\mathbf{\Lambda}[G]$, then G is Class 1.

First, we will show that the lemma follows directly from the claim. Let G be a critical simple graph and uv be any edge of G. By the definition of critical graphs, it is known that $\chi'(G - uv) \leq \Delta$. We also know that $\Delta(G - uv) = \Delta(G)$, since otherwise every vertex of $G - uv$ would miss a color and, thus, we would be able to construct an edge-coloring for G using $\Delta(G)$ colors by Vizing's recoloring procedure. For the sake of contradiction, suppose that the number of vertices in $\mathbf{\Lambda}[G]$ adjacent to u in G is at most:

$$\begin{aligned} \Delta - d_G(v), &\quad \text{if } d_G(v) < \Delta; \\ 1, &\quad \text{if } d_G(v) = \Delta. \end{aligned}$$

Note that, in the case wherein $d_G(v) = \Delta$, if $v \in V(\mathbf{\Lambda}[G])$ then v is the only major in $\mathbf{\Lambda}[G]$ adjacent to u in G. Also note that, in both cases, we have that u is adjacent in $G - uv$ to at most $\Delta - d_G(v)$ vertices in $\mathbf{\Lambda}[G]$. Ergo, by the claim, G is Class 1, but this is a contradiction, because G is critical graph, thus Class 2 by definition.

Now, we will prove the claim. For this, consider an edge-coloring of $G - uv$ using Δ colors.

If $d_G(v) = \Delta$, then u is adjacent in $G - uv$ to no majors in $\mathbf{\Lambda}[G]$. Therefore, all majors adjacent to u in $G - uv$ are strictly non-proper and edge uv is not colored, so we can apply Lemma 4 to obtain an edge-coloring of G using Δ colors.

If $d_G(v) < \Delta$, we know that the numbers of colors missing at u is at least one, and at $v := v_0$ is exactly $\Delta - d_G(v) + 1 \geq 2$. If there is some extended recoloring fan $v_0, v_1, \ldots, v_k$ satisfying any of Properties 3a, 3b, we are done. Otherwise, let $v_0, v_1, \ldots, v_k$ be a maximal recoloring fan for uv. Clearly, v_k is major of G. If $v_k \notin V(\mathbf{\Lambda}[G])$, then it is always possible to recolor some edges of G either to obtain a color virtually missing at v_k, or a complete recoloring fan for uv [5]. Therefore, we assume without loss of generality that for every maximal recoloring fan $v_0, v_1, \ldots, v_k$ for uv, vertex $v_k \in V(\mathbf{\Lambda}[G])$.

For each of the $\Delta - d_G(v) + 1$ missing colors at v_0, we have a distinct maximal recoloring fan starting at v_0. Since u is adjacent to at most $\Delta - d_G(v)$ vertices in $\mathbf{\Lambda}[G]$, we have, by the Pigeonhole Principle, two distinct maximal recoloring fans $v_0, x_1, , \ldots, x_m$ and $v_0, y_1, \ldots, y_n$ such that $x_i = y_j$ for some $i \in \{2, \ldots, m\}$ and some $j \in \{2, \ldots, n\}$. Without loss of generality, we can assume that $x_{i'} \neq y_{j'}$ for all $i' \in \{1, \ldots, i-1\}$ and for all $j' \in \{1, \ldots, j-1\}$.

Let α be the color of $ux_i = uy_j$ and let β be some color missing at u. Initially, uncolor edge ux_i. Now u misses α and, since both x_{i-1} and y_{j-1} miss α, we have two distinct complete fans $v_0, x_1, \ldots, x_{i-1}$ and $v_0, y_1, \ldots, y_{j-1}$. Choose one of them to perform the decay of the colors, say $v_0, x_1, \ldots, x_{i-1}$.

We know that v_0 misses at least two colors, thus, after the decay, we can rearrange the fans to obtain the fan

$$x_i, x_{i-1}, \ldots, x_2, x_1, v_0, y_1, y_2, \ldots, y_{j-2}, y_{j-1}$$

which is a recoloring fan for ux_i satisfying Property 3b, since both x_i and y_{j-1} miss α. $\square$

3. Final remarks

Many results in the literature concerning edge-coloring are derived from Vizing's Adjacency Lemma [4, 6]. With the use of the recent extended recoloring procedure presented in [5], we sought to prove a stronger form of Vizing's Adjacency Lemma. Similarly, we aim to investigate if, through the extended recoloring procedure and the extended adjacency lemma proved in this work, we can also achieve stronger forms of other known results.

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