Minimal feedback arc sets containing long paths in few-vertex tournaments

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Introduction

A digraph D is a pair $D = (V, A)$ in which $V = \{v_1, v_2, \ldots, v_n\}$ is a set of *vertices*, and $A = \{e_1, e_2, \ldots, e_m\} \subseteq V \times V$ is a set of *arcs*. For a pair of vertices $v, w \in V(D)$, we write vw to denote the arc from v to w. The *indegree* of v, denoted by $d^-(v)$, is the number of arcs coming in v, and the *outdegree* of v, denoted by $d^+(v)$, is the number of arcs leaving v. We denote by $\delta^+(D)$ the *minimum outdegree* of D, that is, the smallest number of arcs leaving the same vertex.

A *path* in D is a subdigraph $P \subseteq D$ that admits an ordering $v_0 \cdots v_\ell$ of its vertices for which $A(P) = \{v_i v_{i+1} : i = 0, \ldots, \ell - 1\}$. In this case, $v_1, \ldots, v_{\ell-1}$ are the *internal vertices* of P; v_0 is called the *initial vertex* of P; v_ℓ is called the *end vertex* of P. When convenient, we refer to such path as v_0v_ℓ-path. A *cycle* in D is a subdigraph $C \subseteq D$ obtained from a v_0v_ℓ-path by adding the arc $v_\ell v_0$. The *length* of a path or a cycle is its number of arcs.

The following conjecture, is one of the most celebrated open problems on digraphs. We refer the reader to [3] for more details.

Conjecture 1 (Caccetta and Häggkvist [1])**.** Every digraph D contains a directed cycle of length at most $\lceil n/\delta^+(D) \rceil$.

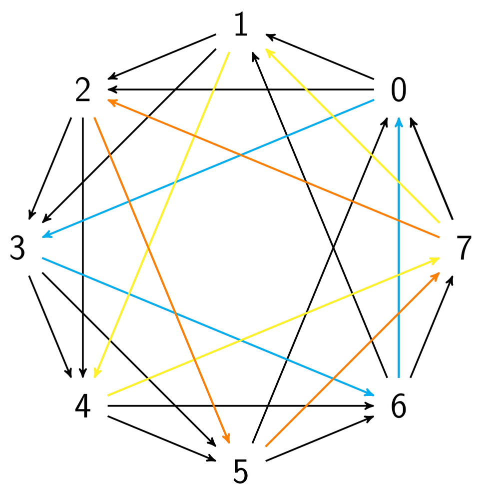

Figure 1: A digraph D for which $|V(D)| = 8$ and $\delta^+(D) = 3$. Some cycles of length 3 are highlighted in blue, yellow and orange.

Lichiardopol's Conjecture

A set of arcs $F \subseteq A(D)$ is a *feedback arc set* (FAS) of D if the digraph obtained from D by removing the edges of F contains no directed cycles. Furthermore, a FAS F is *minimal* if there is no FAS F' for which $F' \subsetneq F$. In this work we explore the following conjecture attributed to Lichiardopol [3]. It's not hard to check that it implies the Caccetta-Häggkvist conjecture.

Conjecture 2 (Lichiardopol)**.** Every digraph D has a minimal FAS that contains a path with length at least $\delta^+(D)$.

In what follows, we call a FAS *good* if it is a minimal FAS that contains a path with length at least $\delta^+(D)$.

Contribution

In this poster we verify Conjecture 2 for tournaments with at most eight vertices.

Note that a digraph D may have a minimal FAS that contains only paths with length smaller than $\delta^+(D)$ while having a good FAS. This situation is illustrated respectively by Figures 2 and 3.

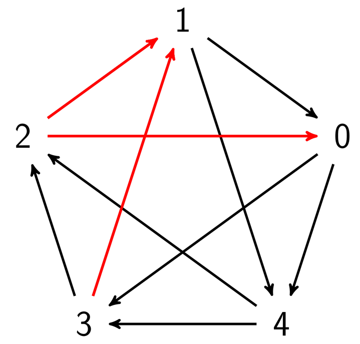

Figure 2: A digraph D for which $\delta^+(D) = 2$. A minimal FAS that contains only paths with length 1 is highlighted in red.

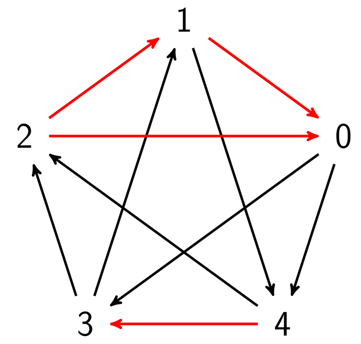

Figure 3: A good FAS highlighted in red.

In what follows, we provide a characterization for minimal feedback arc sets. The *distance* between two vertices $v, w \in V(D)$, denoted by $d_D(v, w)$, is the minimum number of arcs in a vw-path. If there is no vw-path in D, we set $d_D(v, w) = \infty$. A set $A' \subseteq A(D)$ is a *returnable set* if $d_{D-A'}(w, v) < \infty$ for every arc $vw \in A'$, i.e., for every arc $vw \in A(P)$, there is a wv-path, such a path is called *return path* for vw.

Proposition 1. Let $F \subseteq A(D)$ be a FAS. Then, F is a minimal FAS if and only if F is a returnable set.

Proof. If $F \subseteq A(D)$ is a minimal FAS, then for every $vw \in F$, the set $F - vw$ is not a FAS, so $D - (F - vw) = D - F + vw$ has a cycle, and hence $D - F$ contains a wv-path. Therefore, F is a returnable set. On the other hand, if F is a returnable set, then for every arc $vw \in F$, $d_{D-F}(w, v) < \infty$, i.e., there is a wv-path in $D - F$. Thus, suppose that F is a FAS, but not a minimal FAS. In this case, there is an arc $xy \in F$ such that $F' = F - xy$ is a FAS. But the definition of returnable set guarantees that there is a yx-path P_{yx} in $D - F$. Therefore, $P_{yx} \cup xy$ is a cycle in $D - F'$, a contradiction. □

We may reduce the problem of finding a good FAS in a digraph D to finding a good FAS in any subdigraph $D' \subseteq D$ with $\delta^+(D') = \delta^+(D)$ by applying the following lemma repeatedly.

Lemma 1. Let D be a digraph and $e \in A(D)$. If F' is a minimal FAS in $D' = D - e$, then D contains a minimal FAS F such that $F' \subseteq F$.

Proof. If e admits a return path in $D - F'$, then we put $F = F' + e$. By Proposition 1, F is a minimal FAS. Otherwise, F' is a minimal FAS in D and we put $F = F'$. □

Computational Experiments

A *tournament* T is a digraph such that for each pair of vertices $vw \in V(T)$, either vw or wv, but not both, is in $A(T)$. We checked Conjecture 2 to each tournament with at most eight vertices using a binary search algorithm that explores the characterization described in Proposition 1, which presents properties that can be easily tested. To reduce the computational load, we first reduce each tournament D to a digraph $D' \subset D$, for which $d^+_{D'}(v) = \delta^+(D)$, for each $v \in V(D')$. Therefore, if D' has good FAS, then, by Lemma 1, D has a good FAS. This strategy eases some tournaments with small outdegree, but, although successful in our experiments, it is not safe, as explained below.

As noted by one of the referees, it seems that a counterexample to Conjecture 2 was found by Mader in 2006 [2]. Although we could not find this counterexample, we obtained the following construction which also contradicts Conjecture 2. For every $k \geq 3$, let D_k be the digraph obtained from k disjoint arborescences of height at least 3 with minimum outdegree at least k and add an arc from each of the leaves to each of the roots (see Figure 4). The key observation is that every internal vertex of a path in a FAS must have indegree at least 2, but such vertices form an independent set in D_k.

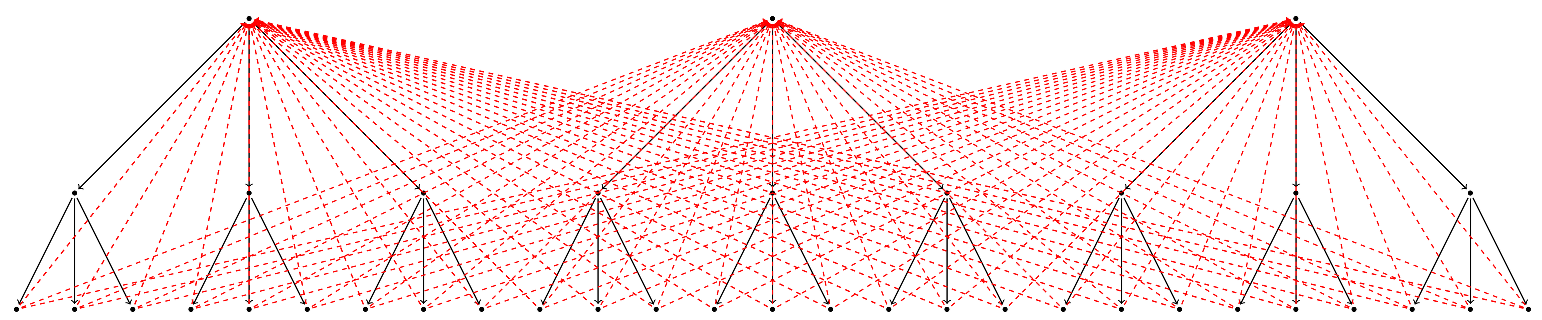

Figure 4: An illustration of a D_3

Conclusion and Future work

Although Conjecture 2 does not hold in its generality, it may still hold for special classes of digraphs as, for example, tournaments. We plan to keep exploring Tournaments to find either a mathematical proof, or a counterexample to Conjecture 2 in this class of digraphs.

References

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