EQUITABLE TOTAL COLORING OF SMALL CUBIC GRAPHS

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1. Introduction

Total coloring is an extremely challenging problem in graph theory. Usually, its applications are to resolve conflict situations. The study of total coloring was boosted in the 1960s, when Behzad and Vizing independently conjectured the *Total Coloring Conjecture*, which gave an upper bound on the number of colors needed to color all elements of the graph without conflict. Later, it was realized that it would be interesting if the number of colored elements with each color were close. Thus, the study of the *equitable total coloring* was started, which is the central theme of this poster, in particular, in small cubic graphs.

2. Basic Definitions

A *simple graph* is an ordered pair $G = (V, E)$, consisting of a non-empty set V of vertices and a set E of edges, defined by pairs of distinct vertices of G. The number of vertices of a graph is represented by n. The degree of a vertex is the number of edges incident to it. A graph is called *regular* when all its vertices have the same degree. A 3-regular graph is called *cubic*.

A *total k-coloring* of a simple graph $G = (V, E)$ assigns at most k colors to the vertices and edges of G such that distinct colors are assigned to every pair of adjacent vertices in V, to every pair of adjacent edges in E, and each vertex and its incident edges. A total coloring is *equitable* if the cardinalities of any two color classes differ by at most 1. The *total chromatic number* $\chi''(G)$ is the minimum k such that G admits a total k-coloring. Behzad and Vizing independently conjectured that for every simple graph, $\Delta + 1 \leq \chi''(G) \leq \Delta + 2$, know as the *Total Coloring Conjecture*, where Δ is the maximum degree in G. If $\chi''(G) = \Delta(G) + 1$ the graph is called *Type 1* and if $\chi''(G) = \Delta(G) + 2$ the graph is called *Type 2*.

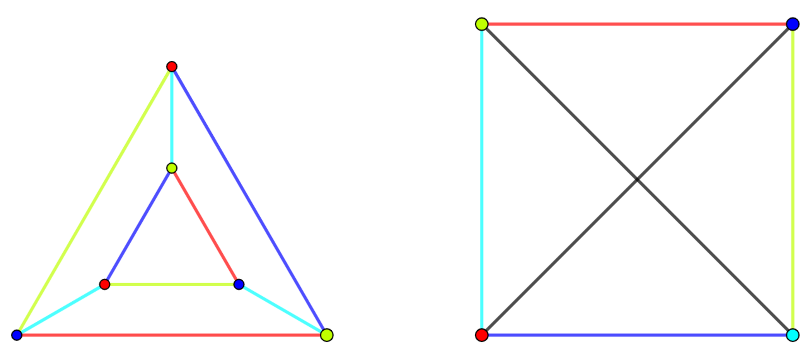

Figure 1: *Optimal total colorings for a Type 1 cubic graph and a Type 2 cubic graph.*

These definitions was obtained from [1] and [5].

3. Stemmock's Conjecture

In 2020, Stemock considered equitable total colorings of cubic graphs in his article [2]. The author conjectured that every total 4-coloring of a cubic graph is equitable if $n < 20$. This upper bound was motivated by a graph with $n = 20$ in the article [4], by Dantas, Figueiredo, Mazzuoccolo, Preissmann, dos Santos and Sasaki, that is Type 1 but does not have an equitable total coloring with 4 colors. This conjecture becomes relevant when we realize that it refers to more than 40000 graphs, which can be verified in the website "House of Graphs".

Cubic Graphs	
Number of vertices	Number of graphs
4	1
6	2
8	5
10	19
12	85
14	509
16	4060
18	41301

After some research in articles about total coloring of cubic graphs, we have noticed that the conjecture is false. In the article [3] we found the total coloring of the circular ladder graphs, denoted by L_n. All graphs in this family, with $n \geq 6$ and n multiple of 3, have a total coloring that is not equitable. Therefore, those that contradict Stemock's conjecture are L_{12} and L_{18}. See the colors below. See Figures 2 and 3 for the colorings. However, we have managed to find the equitable colorings of these two graphs (see Figures 4 and 5) in [4].

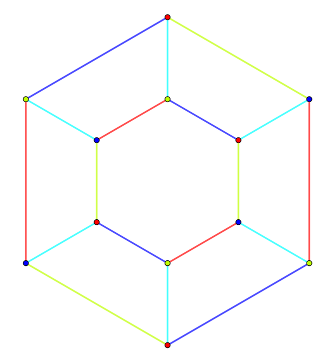

Figure 2: *A total 4-coloring of L_{12}.*

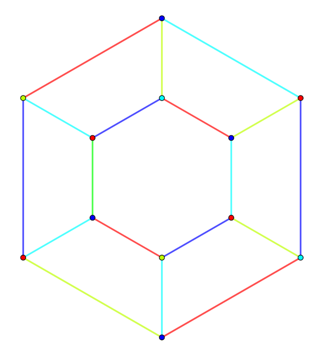

Figure 4: *An equitable total 4-coloring of L_{12}.*

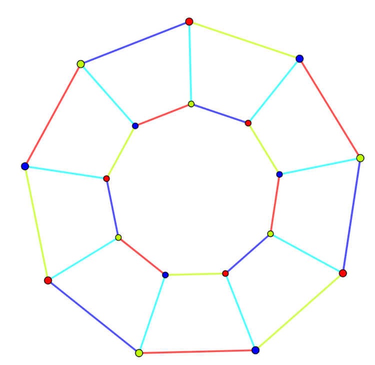

Figure 3: *A total 4-coloring of L_{18}.*

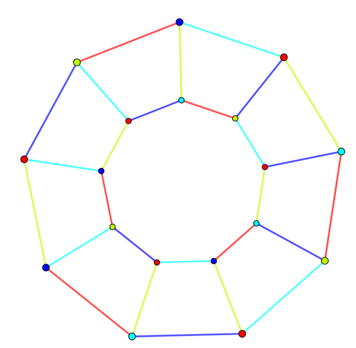

Figure 5: *An equitable total 4-coloring of L_{18}.*

4. Our Approach

Although the conjecture is false for the upper bound it proposes, we raise some interesting questions. This result led us to reformulate Stemock's conjecture as follows.

Conjecture
Every cubic graph, with $n < 20$, that has a total 4-coloring has at least one equitable total 4-coloring.

So far, we have managed to obtain an equitable total 4-coloring for all *Type 1* graphs that we have investigated. Here are some examples.

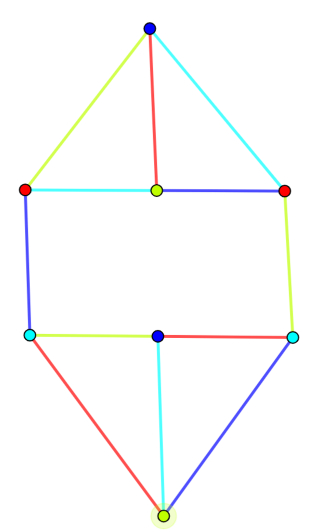

Figure 6: *A cubic graph of 8 vertices with an equitable total 4-coloring.*

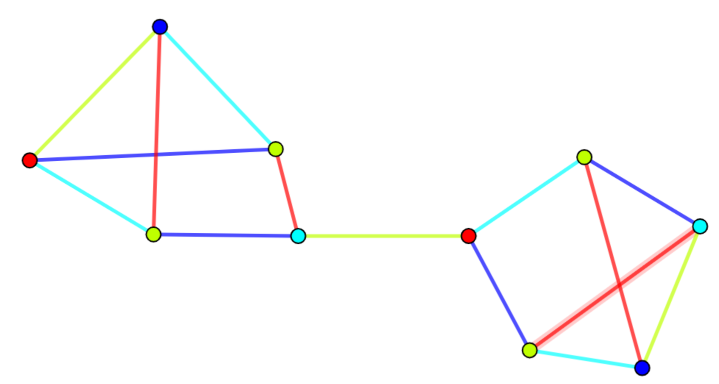

Figure 7: *A cubic graph of 10 vertices with an equitable total 4-coloring.*

5. Our Goals

Moving forward, we have two priority goals:

- Prove the reformulated Stemock's conjecture. If not, given the huge amount of graphs involved, check for cubic graphs with $n \leq 12$.
- Check if, in fact, Stemock's conjecture holds for cubic graphs with $n < 12$.

References

[1] BONDY, J. A and Murty, U. S. R. *Graduate Texts in Mathematics 244 - Graph Theory*. Springer, 2008.

[2] Stemock, B. *On the equitable total $(k+1)$-coloring of k-regular graphs.* Rose-Hulman Undergraduate Mathematics Journal, 2020.

[3] Chetwynd, A. and Hilton, AJW. *Some refinements of the total chromatic number conjecture.* Congressus Numerantium, 1988.

[4] Dantas, S. and Figueiredo, C. and Preissmann, M. and Sasaki, D. and dos Santos, V. *On the equitable total chromatic number of cubic graphs.* Discrete Applied Mathematics, 2016.

[5] SASAKI, D. *Sobre Coloração Total de Grafos Cúbicos.* Rio de Janeiro: UFRJ/COPPE, 2013.

6. Acknowledgment

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