Minimal feedback arc sets containing long paths in few-vertex tournaments

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Keywords: Minimal feedback arc set, Tournament, Lichiardopol's conjecture

A digraph D is a pair $D = (V, A)$ in which $V = \{v_1, v_2, \ldots, v_n\}$ is a set of *vertices*, and $A = \{e_1, e_2, \ldots, e_m\} \subseteq V \times V$ is a set of *arcs*. The *outdegree* (resp. *indegree*) of v, denoted by $d^+(v)$ (resp. $d^-(v)$), is the number of arcs leaving (resp. coming in) v. We denote by $\delta^+(D)$ the *minimum outdegree* of D, that is, the smallest number of arcs leaving a same vertex. A digraph D is *strongly connected* if for each pair $u, v \in V(D)$ there is a directed path from u to v.

A set of arcs $F \subseteq A(D)$ is a *feedback arc set* (FAS) if the digraph obtained from D by removing the arcs in F contains no directed cycles. Furthermore, a feedback arc set F is *minimal* if there is no feedback arc set F' such that $F' \subsetneq F$. In this work, we explore a conjecture attributed to Lichiardopol (see Sullivan, B. D. "A summary of problems and results related to the Caccetta-Häggkvist conjecture" *arXiv preprint math/0605646*, 2006), which says that every digraph D has a minimal feedback arc set containing a path with length at least $\delta^+(D)$.

Lichiardopol's conjecture would imply the well-known Caccetta-Häggkvist, but, as noted by one of the referees, it seems that a counterexample to it was found by Mader in 2006 (Chudnovsky, M. Seymour, P. and Thomas, R. "The Caccetta-Haggkvist conjecture" Workshop, American Institute of Mathematics, San Jose, California, 2006). Although we could not find Mader's specific construction, after the submission of this abstract, we obtained the following counterexample. For every $k \geq 3$, let D_k be the digraph obtained from k disjoint arborescences of height at least 3 with minimum outdegree at least k and add an arc from each of the leaves to each of the roots. The key observation is that every internal vertex of a path in a FAS has indegree at least 2, but such vertices induce an independent set in D_k.

Although Lichiardopol's Conjecture does not hold in its generality, it is still interesting to explore it in specific classes of digraphs. In this work, we explore Lichiardopol's Conjecture in *tournaments*, which are orientations of complete graphs. More specifically by using a binary search algorithm that uses a special characterization of minimal feedback arc sets, we verified its validity to the 6405 strongly connected tournaments with up to eight vertices.

ISSN:2526-1371 DOI:10.5281/zenodo.6090322