On Subclasses of Circular Arc Bigraphs

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Abstract. *Circular arc (CA) bigraphs, a bipartite variation of circular arc graphs, are bipartite graphs such that, for each of its partite sets, there is a one-to-one correspondence between its elements and a family of arcs such that arcs from the opposing families intersect precisely if the corresponding vertices in the graph are neighbors. In this paper, we present results about different subclasses of circular arc bigraphs, including Helly and proper CA bigraphs, as well as circular convex bipartite (CCB) graphs.*

Keywords: Graph classes; Characterization; Helly; Circular arc bigraphs; Circular convex bipartite graphs

1. Introduction

A *bi-circular-arc model* is a triple $(C, \mathbb{I}, \mathbb{E})$ such that C is a circle, and $\mathbb{I}$ and $\mathbb{E}$ are families of arcs over C. The *corresponding graph* of a bi-circular-arc model is constructed by creating a corresponding vertex for every arc in $\mathbb{I} \cup \mathbb{E}$, and an edge between the vertices corresponding to two arcs $I \in \mathbb{I}, E \in \mathbb{E}$ precisely if $I \cap E \neq \emptyset$ [Basu et al. 2013, Groshaus et al. 2019]. A bipartite graph is a *circular arc (CA) bigraph* if and only if it is the corresponding graph of a bi-circular-arc model. If G is the corresponding graph of a bi-circular-arc model, we say that G *admits* said model.

A *proper CA bigraph* is a graph that admits a bi-circular-arc model $(C, \mathbb{I}, \mathbb{E})$ in which $\mathbb{I}$ and $\mathbb{E}$ are *proper families* (i.e. families such that no two elements are properly contained in one another) [Basu et al. 2013]. For a bipartite graph $G = (V, W, E)$, a *biclique* is a maximal subset of $V \cup W$ that induces a bipartite-complete graph. A *Helly CA bigraph* is a graph that admits a bi-circular-arc model $(C, \mathbb{I}, \mathbb{E})$ such that, for every biclique K in the graph, there is a point $p_K \in C$ such that every arc in $\mathbb{I} \cup \mathbb{E}$ that corresponds to a vertex of K contains p_K. We call a model with that property a *Helly bi-circular-arc model*.

Figure 1 contains examples of Helly and Proper CA bigraphs, and their respective bi-circular-arc models. When representing a model $(C, \mathbb{I}, \mathbb{E})$ graphically, circle C is denoted as a dotted circle, and the arcs of $\mathbb{I}$ and $\mathbb{E}$ are then drawn inside and outside the circle, respectively.

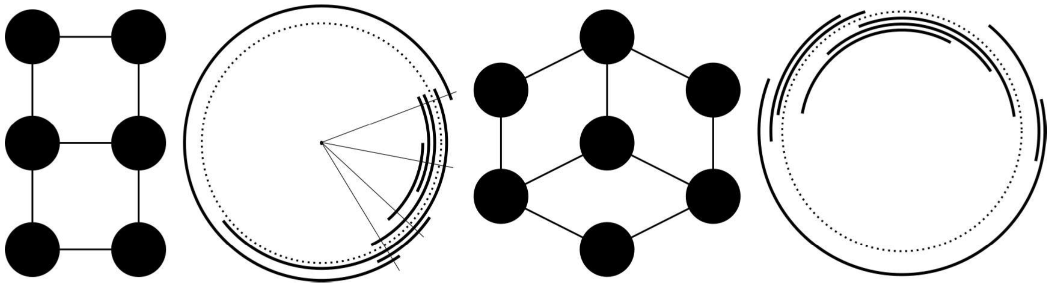

Figure 1. A Helly CA bigraph alongside its Helly bi-circular-arc model (left), and a proper CA bigraph alongside its proper bi-circular-arc model (right). In the Helly model, the points where the bicliques meet are denoted by a line segment starting at the circle's center.

The class of CA bigraphs is a bipartite variation of circular arc graphs [Klee 1969, Lin and Szwarcfiter 2009], and characterizations for the class and its proper, Helly, and unit subclasses exist [Basu et al. 2013, Das and Chakraborty 2015, Groshaus et al. 2019].

A bipartite graph $G = (V, W, E)$ is a *circular convex bipartite (CCB) graph* if V (or W) admits a circular order such that, for every vertex $w \in W$ (or $v \in V$), the neighborhood of w (or v) is consecutive in the given order [Liang and Blum 1995]. We call such an order a *circular convex bipartite order*. The class has been extensively studied for its computational properties, since many hard computational problems can be solved in polynomial time within it [Liang and Blum 1995, Liu et al. 2014]. However, the recognition problem for the class, as well as its relationship with CA bigraphs, have not been extensively studied at the time of this writing. In this paper, we show that CCB graphs are indeed CA bigraphs.

We call a CCB graph $G = (V, W, E)$ *doubly CCB* if both V and W admit CCB orders.

In this article, we present results on CCB graphs, doubly CCB graphs, and proper and Helly CA bigraphs, including containment relations and characterizations. Due to space constraints, we limit ourselves to presenting short ideas of the proofs of every claim, rather than their full proofs.

Given a vertex v, denote by $a(v)$ the arc corresponding to v in a bi-circular-arc model, and for any arc A, denote by $s(A)$ and $t(A)$ its counter-clockwise and clockwise endpoints, respectively. All classes presented remain the same regardless of whether arcs are considered closed or open, but in order to simplify our proofs, we consider every arc to be open unless otherwise stated.

2. Results

We start by presenting a characterization of CCB graphs which we use in some proofs. For that, we introduce what we call *arc-point models*. An arc-point model is a triple $(C, \mathbb{A}, P)$ such that C is a circle, $\mathbb{A}$ is a family of arcs on C and P is a set of points in C. The corresponding graph of an arc-point model is constructed by creating one vertex for every arc in $\mathbb{A}$ and point in P, and for $A \in \mathbb{A}$ and $p \in P$, an edge between the

corresponding vertices of A and p exists precisely if $p \in A$.
Lemma 1. *A graph is a CCB graph if and only if it is the corresponding graph of an arc-point model.*

Proof Idea. ($\Leftarrow$) The clockwise order of the points of P around C is a circular convex bipartite order. ($\Rightarrow$) For the part that admits a CCB order, distribute points around C according to the order. For the opposite part, it is easy to build a family of arcs. □

Applying Lemma 1, we can prove the following theorem.
Theorem 1. *Every CCB graph is a CA bigraph.*

Proof Idea. By replacing the set P of points in an arc-point model with a family of $|P|$ sufficiently small arcs in C, we obtain a bi-circular-arc model with the same corresponding graph. Therefore, CCB graphs are CA bigraphs. □

The following is also an important preliminary result. We call vertices with the same open neighborhood *twins*, and the *twin-free version* of a graph the graph that results from removing one vertex from every pair of twins until no twins remain.
Lemma 2. *A graph is a CCB (doubly CCB) graph if and only if its twin-free version is.*

In the following theorem, we present a pair of containment relations between CCB graphs and subclasses of CA bigraphs.
Theorem 2. *If G is a Helly or proper CA bigraph, then G is a doubly CCB graph.*

Proof Idea. For the Helly case, consider a twin-free graph G that admits a Helly model $(C, \mathbb{I}, \mathbb{E})$. For every vertex v in the part represented by $\mathbb{E}$, there is a biclique K_v containing its closed neighborhood, and since the model is Helly, there is a point $P_{K_v} \in C$ contained in every arc corresponding to elements of K_v. Therefore, we can build an arc-point model $(C, \mathbb{I}, P)$ where P contains the points P_{K_v} for every v in the part represented by $\mathbb{E}$.

For the proper case, consider a model $(C, \mathbb{I}, \mathbb{E})$, with $\mathbb{I}$ and $\mathbb{E}$ proper, corresponding to twin-free graph G. Consider the circular order in which the counter-clockwise endpoints of $\mathbb{I}$ are encountered in a clockwise traversal of the circle. That order is a CCB order of the part represented by $\mathbb{I}$.

Both classes are doubly CCB, since the proofs can be applied equally to both partite sets. □

The converse of Theorem 2 is not true, and Figure 2 contains counter-examples for both cases.

The following result presents a characterization of proper CA bigraphs within the class of CCB graphs. In a bipartite graph $G = (V, W, E)$, we call a vertex $v \in V$ ($w \in W$) *bi-universal* if $N(v) = W$ ($N(w) = V$).
Lemma 3. *A bipartite graph $G = (V, W, E)$ without bi-universal vertices is a proper CA bigraph if and only if V admits a CCB order such that, for every pair of elements $w_1, w_2 \in W$ satisfying $N(w_1) \subset N(w_2)$, the intervals of the CCB order corresponding to the vertices in $N(w_1)$ and $N(w_2)$ either begin or end in the same vertex.*

Proof Idea. Theorem 3.6 from [Basu et al. 2013] states that a bipartite graph if a proper CA bigraph if and only if it admits a biadjacency matrix with a *monotone circular arrangement* (MCA). The order of the columns on a matrix with an MCA is a CCB order with the properties stated in the lemma.

Conversely, if a graph admits a CCB order with the properties stated in the lemma, it is easy to show that there exists a biadjacency matrix with an MCA such that the order of columns is given by the CCB order. □

We close our results section presenting a containment relation between a subclass of Helly CA bigraphs and proper CA bigraphs. A bipartite graph is *bichordal* if it does not contain an induced cycle of order greater than 4. The following characterization, presented without proof due to space constraints, is applied in the containment relation.

Lemma 4. *Consider graph G, with $V(G)$ being partitioned in the following subsets, for $k \geq 6, n_1, ..., n_k \geq 0$:*

- $C = \{c_1, c_2, ..., c_k\}$.
- $V = \{v_1, v_2, ..., v_k\}$.
- $W_i = \{w_{i,1}, ..., w_{i,n_i}\}$ *for all* $1 \leq i \leq k$.
- $X_i = \{x_{i,1}, ..., x_{i,n_i}\}$ *for all* $1 \leq i \leq k$.

Let the neighborhoods of the vertices in $V(G)$ be the following, for all $1 \leq i \leq k$. Consider cyclic summation (e.g. $1 - 1 = k, k + 1 = 1$) for vertex and set indices:

- $N(c_i) = \{c_{i-1}, c_{i+1}, v_i\} \cup W_i \cup X_{i-1}$.
- $N(v_i) = \{c_i\}$.
- $N(w_{i,j}) = \{c_i\} \cup \{x_{i,l} \in X_i | l \leq j\}$, *for all* $1 \leq j \leq n_i$.
- $N(x_{i,j}) = \{c_{i+1}\} \cup \{w_{i,l} \in W_i | l \geq j\}$, *for all* $1 \leq j \leq n_i$.

A bipartite graph that is not bichordal is a Helly CA bigraph if and only if its twin-free version is an induced subgraph of G, for certain values of $k, n_1, ..., n_k$.

Theorem 3. *If G is a Helly CA bigraph that is not bichordal, then G is a proper CA bigraph.*

Proof Idea. It suffices to prove that graph G from Lemma 4 is a proper CA bigraph.

Note that G has no bi-universal vertices, so we can apply Lemma 3 to prove that G is a proper CA bigraph.

The partite sets of G are the following:

- $X = C_1 \cup V_2 \cup \bigcup_{i=2,i+2}^{k} W_i \cup \bigcup_{i=1,i+2}^{k-1} X_i$.
- $Y = C_2 \cup V_1 \cup \bigcup_{i=2,i+2}^{k} X_i \cup \bigcup_{i=1,i+2}^{k-1} W_i$.

Consider, then, the following circular ordering of X:

$$(c_1, x_{1,1}, ..., x_{1,n_1}, v_2, w_{2,1}, ..., w_{2,n_2}, c_3, ...$$
$$..., c_{k-1}, x_{k-1,1}, ..., x_{k-1,n_{k-1}}, v_k, w_{k,1}, ..., w_{k,n_k})$$

The above order, in simple terms, is built as follows: for every i (starting at $i = 1$), add vertex c_i, followed by the elements of X_i in ascending order, followed by v_{i+1}, followed by W_{i+1} in ascending order, and then the loop restarts with $i + 2$ until $i > k$.

Note that the presented order is a CCB order. Furthermore, the only proper containments between neighborhoods of Y are the following, for $i \leq k, l_1, l_2 \leq n_i$:

- Between pairs of the form x_{i,l_1}, x_{i,l_2}.
- Between pairs of the form w_{i,l_1}, w_{i,l_2}.
- Between vertices of W_i and c_{i+1}.
- Between vertices of X_i and c_i.
- Between v_i and any vertex from W_i, X_{i-1}, or $\{c_{i-1}, c_{i+1}\}$.

In the given CCB order of X, all pairs listed above are such that the intervals containing their neighborhoods have an endpoint in common. Therefore, G is a proper CA bigraph.

Note that, if a graph is a proper CA bigraph, the addition of twins to it makes it remain a proper CA bigraph. To add a twin arc to a family without creating proper containments, just copy the original arc and shift the endpoints a significantly small amount. Therefore, we conclude that every Helly CA bigraph that is not bichordal is a proper CA bigraph. □

Figure 2 is a Venn diagram of the classes mentioned in the paper. The regions marked as *CAB, CCB*, and *D-CCB* represent, respectively, CA bigraphs, CCB graphs and doubly CCB graphs, while the regions marked as *P, H* and *NBH* represent proper and Helly CA bigraphs, and Helly CA bigraphs that are not bichordal, respecetively.

3. Conclusion

We presented a characterization of circular convex bipartite graphs using the concept of arc-point models. We showed that doubly CCB graphs, a proper subclass of circular convex bipartite graphs, are a proper superclass of both Helly and proper circular arc bigraphs. We also showed a result that proves that non-bichordal Helly circular arc bigraphs are a proper subclass of proper circular arc bigraphs, and presented a characterization of proper CA bigraphs without bi-universal vertices within the class of CCB graphs.

Future work includes searching for recognition algorithms for circular convex bipartite graphs, and studying other potential characterizations of proper and Helly CA bigraphs with relation to the class of CCB graphs.

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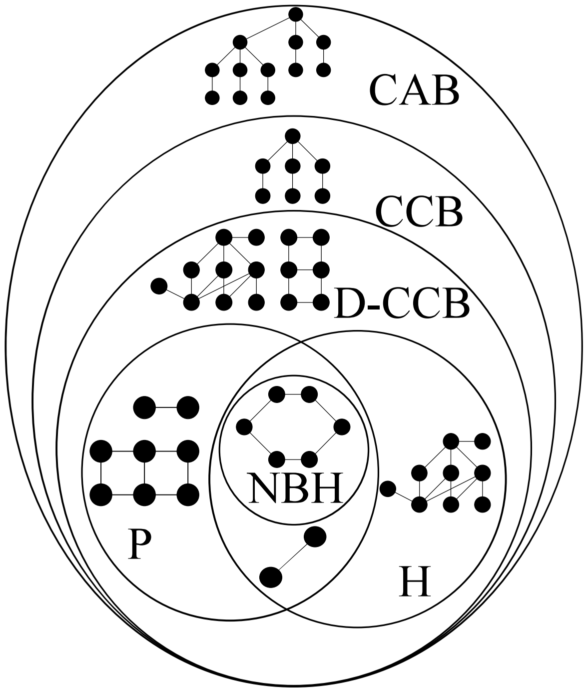

Figure 2. A Venn diagram of the classes mentioned in the paper, with an example graph in each region.

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