

On conformability of line graphs of complete graphs

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Abstract

A Δ -regular graph G is conformable if it has a $(\Delta + 1)$ -vertex coloring where the cardinality of each vertex color class has the same parity as the order of the graph. A general characterization for conformable graphs is unknown. The importance of conformability is due to the fact that it can be an auxiliary tool toward determining the total chromatic number of a regular graph. Being conformable is a necessary condition for a graph to be *Type 1*. In this paper, we show a positive evidence to the conjecture proposed in 2018 which states that all line graphs of complete graphs $L(K_n)$ are *Type 1*, by proving that they are all conformable.

Keywords: conformable, total coloring, complete graph, line graph

1. Introduction

Let $G = (V, E)$ be a simple connected graph. A k -total coloring of G is an assignment of k colors to the vertices and edges of G so that adjacent or incident elements have different colors. The total chromatic number of G , denoted by $\chi''(G)$, is the smallest k for which G has a k -total coloring. Clearly, $\chi''(G) \geq \Delta(G) + 1$ and the Total Coloring Conjecture (TCC) states that the total chromatic number of any graph is at most $\Delta(G) + 2$, where $\Delta(G)$ is the maximum degree of the graph [1, 2]. If the TCC holds, graphs with $\chi''(G) = \Delta(G) + 1$ are called *Type 1*, and graphs with $\chi''(G) = \Delta(G) + 2$ are called *Type 2*. In 1989, Sánchez-Arroyo [3] proved that determining the total chromatic number of an arbitrary graph is a NP-hard problem.

A *Type 1* graph has a nice structural property due to Chetwynd and Hilton [4]. Let the deficiency of G be $def(G) = \sum_{v \in V(G)} (\Delta(G) - d_G(v))$, where $d_G(v)$ is the degree of a vertex v in G . A vertex coloring $\varphi : V(G) \rightarrow \{1, 2, \dots, \Delta(G) + 1\}$ is called conformable if the number of color classes (including empty color classes) of parity different from that of $|V(G)|$ is at most $def(G)$. Note that since regular graphs have deficiency 0, each color class has the same parity as $|V(G)|$. A graph is conformable if and only if it has a conformable vertex coloring, since otherwise it is called non-conformable.

It is well known that every *Type 1* graph is conformable [4]. In the same paper, Chetwynd and Hilton conjectured that for graphs with maximum degree greater than one half of their order the converse is essentially true. However, in some cases this is not true: For instance,

the complete bipartite graphs $K_{n,n}$, for even $n > 1$, and the Möbius ladders M_{2n} , for $n > 3$, are conformable and *Type 2* [4]. Determining suitable conformable vertex colorings or proving the non-conformability of a graph G are useful tools to determine the total chromatic number of a graph. Indeed, every non-conformable graph is *Type 2* and suitable conformable vertex colorings may be extended to $(\Delta(G) + 1)$ -total colorings.

There are few results on conformability, such as a characterization of non-conformability of some graphs and the verification of Chetwynd and Hilton's conjecture for special classes of graphs [5, 6]. In this paper, we prove that the line graphs of complete graphs $L(K_n)$ are conformable, contributing to the conjecture proposed by Vignesh et al. [7] in 2018 that all line graphs $L(K_n)$ are *Type 1*. Note that the small cases $L(K_3) \simeq K_3$ and $L(K_4) \simeq C_6^2$ are trivial examples of *Type 1* graphs.

2. Main Result

In this section, we prove that all graphs $L(K_n)$ are conformable. Note that $V(L(K_n)) = \frac{n(n-1)}{2}$, and first we mention a useful lemma.

Lemma 1. *If G is regular of degree k , then $L(G)$ is regular of degree $2k - 2$.*

Consequently, the degree of $L(K_n)$ is $2(n - 1) - 2 = 2n - 4$ and so in order to prove that $L(K_n)$ is conformable, we exhibit a vertex coloring with $\Delta(L(K_n)) + 1 = 2n - 3$ colors such that the number of vertices in each color class has the same parity as $|V(L(K_n))|$.

Theorem 1. *The graphs $L(K_n)$ are conformable.*

Proof. First consider $V(K_n) = \{v_0, v_1, \dots, v_{n-1}\}$. By definition of line graphs, each edge of K_n is identified with a vertex in $L(K_n)$.

The proof is separated into two cases as follows:

- For n odd: let $n = 2k - 1$, the number of vertices of $L(K_n)$ is given by $\frac{n(n-1)}{2} = (2k - 1)(k - 1)$. So, we consider the cases where k is odd and k is even, separately.

If k is odd, then the number of vertices of $L(K_n)$ is even. Therefore, for each $1 \leq p \leq n$, we will define the set C_p as the set of vertices of $L(K_n)$ with color p . This is given by $C_p = \{v_{p-1-q}v_{p-1+q} \mid 1 \leq q \leq \frac{n-1}{2}\}$, with $v_i \in V(K_n)$ where the indexes of the vertices are considered modulo n . Note that C_p is a maximum matching in K_n , and so it is a maximal independent set on $L(K_n)$. Moreover, $\bigcup_{i=1}^n C_i = E(K_n) = V(L(K_n))$. Consequently this vertex coloring is conformable, because $|C_p| = \frac{n-1}{2} = k - 1$ is even and we use $\Delta(L(K_n)) + 1$ colors (where $n - 3$ color classes are empty). Figure 1a shows an example of the vertex coloring for this case.

If k is even, then the number of vertices of $L(K_n)$ is odd. We use C_p as before, for $1 \leq p \leq n - 2$. The vertices $v_{n-2-q}v_{n-2+q}$ and $v_{n-1-q}v_{n-1+q}$ will be assigned with colors $n - 1 + q - 1$ and $n + q - 1 + \frac{n-1}{2} - 1$, respectively for each $1 \leq q \leq \frac{n-1}{2}$. As

$|C_p| = \frac{n-1}{2} = k - 1$ is odd, the colors assigned to $v_{n-2+q}v_{n-2-q}$ and $v_{n-1+q}v_{n-1-q}$ appear only once and we use $n - 2 + \frac{n-1}{2} + \frac{n-1}{2} = 2n - 3 = \Delta(L(K_n)) + 1$ colors. So, we conclude that this vertex coloring is conformable. Figure 1b shows an example of the vertex coloring for this case.

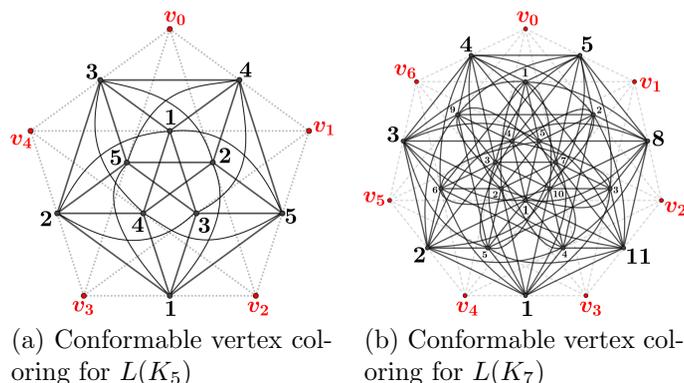


Figure 1: A depiction of a conformable vertex coloring for $L(K_n)$, with n odd.

- For n even: let $n = 2k$, note that the number of vertices of $L(K_n)$ is given by $\frac{n(n-1)}{2} = k(2k - 1)$. Therefore, we consider the cases where k is odd and k is even, separately.

If k is odd, then the number of vertices of $L(K_n)$ is odd. So, we define the set A_p as the set of vertices of $L(K_n)$ with color p . It is given by $A_p = \{v_{p-q}v_{p-1+q} \mid 1 \leq q \leq \frac{n}{2}\}$, with $1 \leq p \leq \frac{n}{2}$. The Figure 2 shows the set A_p for $L(K_{10})$ by edge coloring of K_{10} .

Note that A_p is a maximum matching of K_n for each p . Moreover, $K_n \setminus \bigcup_{p=1}^{\frac{n}{2}} A_p$ are two copies of $K_{\frac{n}{2}} \simeq K_k$. That is, the subgraph induced by the vertices of odd indexes is the graph K_k , as well as the subgraph induced by the vertices of even indexes is the other graph K_k . Let $u_i = v_{2i}$ and $w_i = v_{2i+1}$, with $0 \leq i \leq \frac{n}{2} - 1$. Consider the set of color $j = \frac{n}{2} + p$ as $B_j = \{u_{j-1-q}u_{j-1+q} \mid 1 \leq q \leq \frac{k-1}{2}\} \cup \{w_{j-1-q}w_{j-1+q} \mid 1 \leq q \leq \frac{k-1}{2} - 1\}$, with $1 \leq p \leq k - 1$ and the indexes of vertices are considered modulo k . Figure 2 shows an example of the construction of a vertex coloring using B_j for $L(K_{10})$. The remaining $2k - 2$ uncolored edges of each K_k will receive different colors from the ones used until now, i.e., each one of these edges is an element of a different color class.

We used $k + k - 1 + 2k - 2 = 4k - 3 = 2n - 3 = \Delta(L(K_n)) + 1$ colors. Moreover, the parity of each class of color $|A_p| = k$ is odd as well as $|B_j| = \frac{k-1}{2} + \frac{k-1}{2} - 1 = k - 2$, and finally the remaining color classes, which have only one element, have parity odd. Therefore, we conclude that this vertex coloring is conformable.

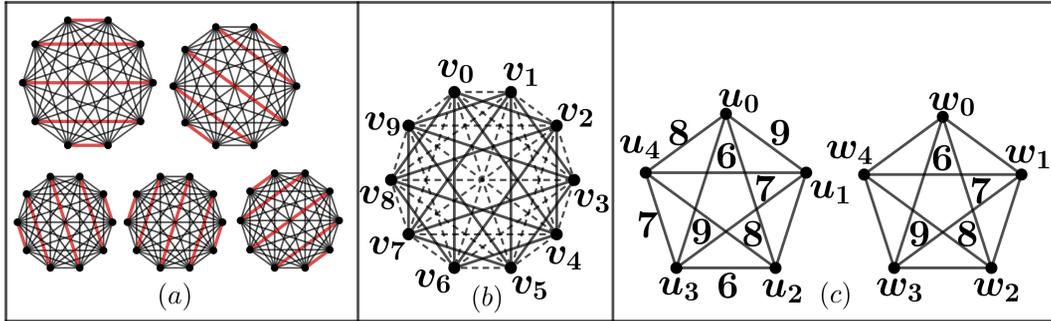


Figure 2: (a) The set A_p to K_{10} for each $1 \leq p \leq 5$, represented by edge with red color; (b) the induced subgraph that are two copies of K_5 ; (c) and the set B_j to 2 copies of K_5 , for each $6 \leq j \leq 9$.

If k is even, then the number of vertices of $L(K_n)$ is even. First we consider A_{p_0} , defined as above with $1 \leq p_0 \leq \frac{n}{2}$. The Figure 3 shows the set A_{p_0} for $L(K_8)$ by edge coloring of K_8 . Similarly, $K_n \setminus \bigcup_{p_0=1}^{\frac{n}{2}} A_{p_0}$ results in two copies of K_k . As k is even, so $k = 2k_1$. If k_1 is even we can repeat, recursively, this process again, using $A_{\frac{n}{2}+p_1}$ in each K_{k_1} , where $1 \leq p_1 \leq \frac{n}{4}$ (Figure 3) and end when for some k_j we have $K_{k_j} \setminus \bigcup_{p_j=1}^{\frac{n}{2^j}} A_{\sum_{i=1}^{j-1} \frac{n}{2^i} + p_j} = \emptyset$. If at any time we have k_j odd, then we apply the coloring C_ℓ , with $\ell = (\frac{n}{2} + \frac{n}{4} + \dots + \frac{n}{2^j}) + p_j$ and $1 \leq p_j \leq k_j - 1$, for each copy K_{k_j} . Note that, for any color λ used in A_λ , $|A_\lambda|$ is even and in every process $|C_\ell|$ is used for even copies. Therefore, the parity of each color class is the same to the number of vertices and we use at most $\frac{\binom{n}{2} (1 - (\frac{1}{2})^{\lceil \log_2 n \rceil})}{1 - \frac{1}{2}} = n \left(1 - (\frac{1}{2})^{\lceil \log_2 n \rceil} \right)$ colors, with $n \left(1 + (\frac{1}{2})^{\lceil \log_2 n \rceil} \right) - 3$ color classes with zero occurrences. We conclude that this vertex coloring is conformable.

□

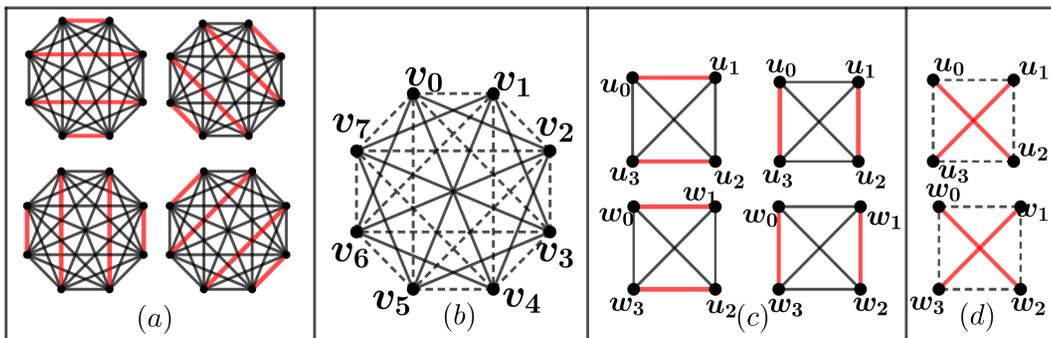


Figure 3: (a) The set A_{p_0} to K_8 , for each $1 \leq p_0 \leq 4$, represented by edge with red color; (b) the induced subgraph that are two copies of K_4 ; (c) the set $A_{\frac{n}{2}+p_1}$ to 2 copies of K_4 , for each $5 \leq p_1 \leq 6$; (d) and the set $A_{\frac{n}{2}+\frac{n}{4}+p_2}$ to 4 copies of K_2 , for $p_2 = 7$.

3. Conclusion

In this paper, we proved that all line graphs $L(K_n)$ are conformable. Note that we can easily extend our conformable vertex coloring of $L(K_5)$ to a total coloring of this graph, showing that $L(K_5)$ is *Type 1* (Figure 4). This is a single and preliminar evidence that the conjecture proposed by Vignesh et al. [7] is valid and we will continue to investigate this conjecture.

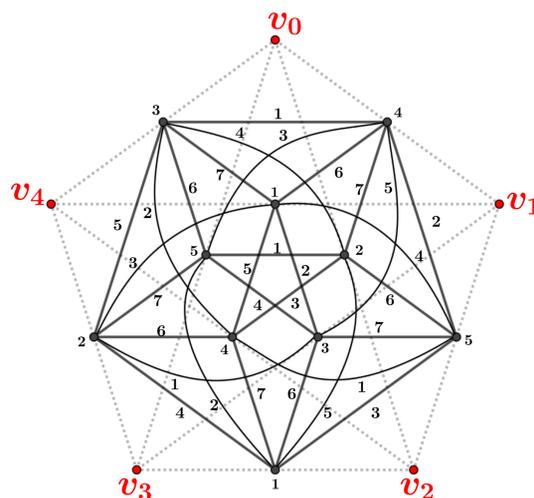


Figure 4: A $(\Delta(L(K_5)) + 1)$ -total coloring to $L(K_5)$.

References

- [1] M. Behzad, Graphs and their chromatic numbers, Ph.D. thesis, Michigan State University, 1965.
- [2] V. Vizing, On an estimate of the chromatic class of a p-graph, *Metody Diskret. Analiz.* (1964) 25–30.
- [3] A. Sánchez-Arroyo, Determining the total colouring number is NP-hard, *Discrete Math.* (1989) 315–319.
- [4] A. G. Chetwynd, A. J. W. Hilton, Some refinements of the total chromatic number conjecture, *Congr. Numer.* (1988) 195–216.
- [5] G. M. Hamilton, A. J. W. Hilton, H. R. F. Hind, Totally critical even order graphs, *J. Comb. Theory Ser. B.* 76 (1999) 262–279.
- [6] A. J. W. Hilton, H. R. Hind, Non-conformable subgraphs of non-conformable graphs, *Discrete Math.* (2002) 203–224.
- [7] R. Vignesh, J. Geetha, K. Somasundaram, Total coloring conjecture for certain classes of graphs, *Algorithms* 11 (2018).