$\mathcal{P} = \mathcal{NP}$ or 5-Snarks Exist

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Abstract. *Similar to the way snarks are defined for* 3*-regular graphs, we present the definition of* 5-snarks *for* 5*-regular graphs. Although we do not know any such graph yet, we prove that they must exist, unless* $\mathcal{P} = \mathcal{NP}$*. We also prove that, unless* $\mathcal{P} = \mathcal{NP}$*, the number of* 5*-snarks of a given order cannot be polynomially bounded.*

Resumo. *De modo similar a como snarks são definidos para grafos* 3*-regulares, apresentamos a definição de* 5-snarks *para grafos* 5*-regulares. Embora não conheçamos nenhum destes grafos ainda, provamos que eles precisam existir, a menos que* $\mathcal{P} = \mathcal{NP}$*. Também provamos que, a menos que* $\mathcal{P} = \mathcal{NP}$*, o número de* 5*-snarks de uma dada ordem não pode ser limitado polinomialmente.*

Keywords: Graph theory in relation to Computer Science (MSC 68R10); Computational difficulty of problems (MSC 68Q17); Colouring of graphs and hypergraphs (MSC 05C15).

1. Introduction

The *Four Colour Theorem* is a major and well-known result in the history of Mathematics which states that *no more than four colours suffice to colour any map in such a way that no two adjacent regions are coloured the same*. This statement was first conjectured in 1852 and it took more than 120 years to be proved [Appel and Haken 1977, Appel et al. 1977]. During the 20th century, many research branches in Combinatorics and Computer Science have been developed from the efforts on trying to prove the Four Colour Theorem, specially the studies on graph colouring problems and on graph planarity and embeddings, leading to several important applications [Fu and Ma 2013, Lewis 2016].

Amongst the earliest results on the Four Colour Problem is Tait's Theorem, which states that the Four Colour Theorem (Conjecture, by that time) is equivalent to the statement that there is no cubic (i.e. 3-regular) 2-edge-connected non-3-edge-colourable graph[1] which is planar [Tait 1880]. Cubic 2-edge-connected non-3-edge-colourable graphs have very peculiar properties and play an important role concerning many graph problems [Chladný and Škoviera 2010]. These graphs were named *snarks* by [Gardner 1976] after the mysterious thing hunted in the poem *The Hunting of the Snark*, by Lewis Carroll.

The first snark to be known was the Petersen graph (Fig. 1a), which, although credited to Petersen [Petersen 1898], had already appeared earlier [Kempe 1886]. It took

[1]We refer the reader to [Diestel 2010] for graph-theoretical definitions, whose notation we follow. For definitions on Computational Complexity, we refer the reader to [Arora and Borak 2007].

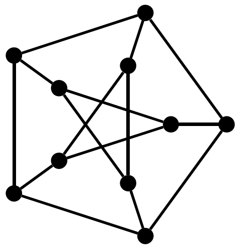
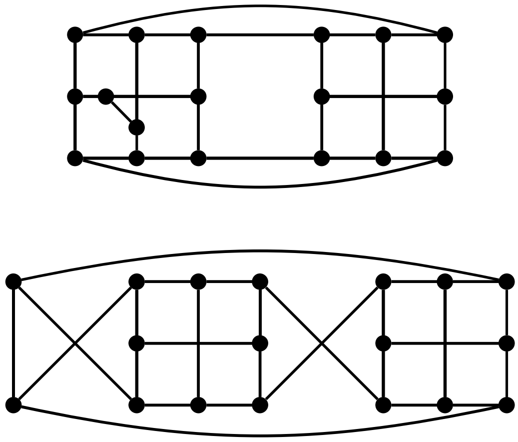
a)  b) 

Figure 1. The Petersen graph (a) and the Blanuša snarks (b)

half a century for other three snarks to be found: the two 18-vertex Blanuša snarks [Blanuša 1946] (Fig. 1b) and the 210-vertex Descartes snark [Descartes 1948]. The first infinite family of snarks to be shown are the *Flower snarks* [Isaacs 1975]. Now it is known that the number of snarks of sufficiently large even order n is at least $2^{(n-84)/18}$ [Skupień 2007] (recall that there is no cubic graph of odd order). However, despite this exponential lower bound, snarks can be still regarded as *rare* graphs, since, for every $d \geq 3$, the proportion of d-regular non-d-edge-colourable graphs of even order n over all n-order d-regular graphs goes to 0 as $n/2$ goes to ∞ [Robinson and Wormald 1994].

In this paper, we introduce the 5-*snarks*, defined below for 5-regular graphs similarly as snarks are defined for cubic graphs. As much is known about the relation between snarks and important graph problems and conjectures when restricted to cubic graphs, the aim of investigating 5-snarks is to improve the state of the art of these problems and conjectures concerning graphs of higher degree.

Definition. A 5-*snark* is a 5-regular 4-edge-connected non-5-edge-colourable graph.

Although we do not know yet any graph fitting this definition, we prove that:

Theorem 1. *If* 5-*snarks do not exist, then* $\mathcal{P} = \mathcal{NP}$.

Not only these graphs *must* exist, but there cannot be *too few* of them[2].

Corollary 2. *There cannot be a polynomial* $p(n)$ *which bounds above the number of* 5-*snarks of order* n*, unless* $\mathcal{P} = \mathcal{NP}$.

Amongst the important open graph problems related to the snarks are the *Overfull* and the *1-Factorisation Conjectures* in edge-colouring[3]. A graph G is said to be *subgraph-overfull* (*SO*) if has a subgraph H with $\Delta(H) = \Delta(G) =: \Delta$ which is *overfull*, i.e. $|E(H)| > \Delta\lfloor|V(H)|/2\rfloor$. Clearly, no *SO* graph can be Δ-edge-colourable, but the converse does not hold, and in the same manner that the snarks are examples of cubic graphs which are neither 3-edge-colourable nor *SO*, the 5-snarks are examples of 5-regular

[2]We remark that there cannot also be *too many* 5-snarks, again from [Robinson and Wormald 1994].

[3]We refer the reader to [Zatesko 2018, Ch. 2] for references and details on these important conjectures.

graphs which are neither 5-edge-colourable nor *SO* (see Observation 4). This implies that 5-snarks have at least 16 vertices, unless the *Overfull Conjecture* does not hold, since this conjecture states that all non-Δ-edge-colourable graphs with less than 3Δ vertices are *SO*.

The proofs for our results are presented next. We remark that, in this paper, graphs are assumed to be undirected and loopless, but they are allowed to have parallel edges.

2. Proofs

Consider the following decision problem.

EDGE-COLOURING:
Instance: a graph G;
Question: is G a $\Delta(G)$-edge-colourable[4] graph?

This problem is $\mathcal{NP}$-complete [Holyer 1981] even restricted to d-regular graphs for any constant $d \geq 3$ [Leven and Galil 1983].

Now, for any constant $d \geq 3$, consider this restriction of EDGE-COLOURING.

EDGE-COLOURING(d-regular, $(d-1)$-edge-connected):
Instance: a d-regular $(d-1)$-edge-connected graph G;
Question: is G a d-edge-colourable graph?

In order to prove Theorem 1, we shall first prove the following.

Theorem 3. EDGE-COLOURING(5-regular, 4-edge-connected) *is $\mathcal{NP}$-complete.*

The proof of Theorem 3 goes by reducing EDGE-COLOURING(5-regular) to its restriction EDGE-COLOURING(5-regular, 4-edge-connected). In order to do so using a traditional Karp reduction [Karp 1972], we should present a polynomial-time algorithm which, receiving a 5-regular graph G, outputs a 5-regular 4-edge-connected graph $G^{\ddagger}$ which is 5-edge-colourable if and only if G is 5-edge-colourable. However, our proof uses a Turing oracle reduction[5] (as the reduction used in Cook's proof of the $\mathcal{NP}$-completeness of the Boolean satisfiability problem [Cook 1971]) instead of a Karp reduction. In the context of our problem, such reduction can be viewed as a polynomial-time algorithm which, receiving a 5-regular graph G, outputs not only one, but possibly several 5-regular 4-edge-connected graphs which are *all* 5-edge-colourable if and only if G is 5-edge-colourable (or, equivalently, outputs a possibly disconnected 5-regular graph $G^{\ddagger}$ whose connected components are 4-edge-connected in a manner that $G^{\ddagger}$ is 5-edge-colourable if and only if G is 5-edge-colourable). Observe that the construction of such reduction still implies that if EDGE-COLOURING(5-regular, 4-edge-connected) is in $\mathcal{P}$, then $\mathcal{P} = \mathcal{NP}$.

In the construction of our reduction, we assume that the input of the reduction, i.e. the instance of EDGE-COLOURING(5-regular), has no cut with 1 or 3 edges, in view of an argument on edge-colouring regular graphs known as the *Parity Lemma* [Isaacs 1975].

[4]No graph admits edge-colourings using less than Δ colours. Since every simple graph is $(\Delta+1)$-edge-colourable [Vizing 1964], Δ-edge-colourable graphs and non-Δ-edge-colourable graphs are often referred to as *Class 1* and to as *Class 2* graphs in the literature, respectively. However, we avoid these terms in this paper as the graphs dealt may not be simple (and hence may require more than $\Delta + 1$ colours).

[5]We refer the reader to [Arora and Borak 2007] for a more detailed discussion on types of reductions.

This lemma states that if G is a d-regular d-edge-colourable graph and $F \subseteq E(G)$ is a cut in G, then, for any d-edge-colouring of G with colour set $\{1, \ldots, d\}$,

$$f_1 \equiv f_2 \equiv \cdots \equiv f_d \pmod 2,$$

being f_α the number of edges in F coloured $\alpha \in \{1, \ldots, d\}$. From the Parity Lemma follows that if G is a d-regular graph and has a cut $F \subseteq E(G)$ such that $|F|$ is odd and strictly less than d, then G cannot be d-edge-colourable. Hence, the restriction of EDGE-COLOURING(5-regular) to graphs with no cut with 1 or 3 edges is as hard as EDGE-COLOURING(5-regular) itself, since for graphs having a cut with 1 or 3 edges the answer of the problem is known — it is *no*. Recall that it can be decided in polynomial time if G has a cut with 1 or 3 edges [Ford and Fulkerson 1956, Edmonds and Karp 1972].

Proof of Theorem 3. Let G be a 5-regular graph having no cut with 1 or 3 edges. We shall show how to construct in polynomial time a 5-regular graph $G^\ddagger$ whose connected components are 4-edge-connected in a manner that $G^\ddagger$ is 5-edge-colourable if and only if G is 5-edge-colourable. If G is already 4-edge-connected, then our reduction simply outputs $G^\ddagger := G$. If G is not 4-edge-connected, then all its cuts of cardinality smaller than 4 have 0 or 2 edges. While some connected component of G still has a cut F with 2 edges uv and $u'v'$, being u and u' on the same side of the cut, we replace the edges uv and $u'v'$ with the edges uu' and vv' (parallel edges can be created this way, but they are not a problem), as Fig. 2 illustrates. In the end, the result is a 5-regular graph $G^\ddagger$ whose connected components are 4-edge-connected. Notice that iteratively finding cuts with 2 edges in G until it has none can be done in polynomial time. Also, since we only consider cuts with 2 edges connecting vertices in the same connected component, the number of such cuts in the graph G is decremented at each cut considered, so our reduction halts.

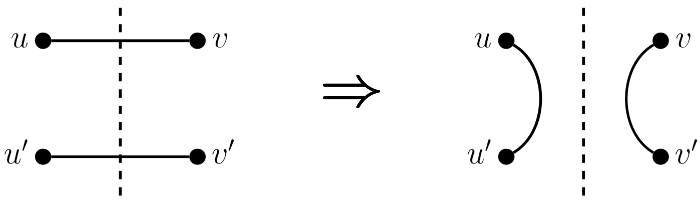


Figure 2. The acting of the reduction over a cut with two edges

We shall prove that G is 5-edge-colourable if and only if $G^\ddagger$ is 5-edge-colourable.

- If G admits a 5-edge-colouring, then, for every cut $F = \{uv, u'v'\}$ considered in an iteration of the reduction, the Parity Lemma guarantees that uv and $u'v'$ are coloured the same. Hence, we can transfer this colour to the edges uu' and vv' and this leads to the construction of a 5-edge-colouring of $G^\ddagger$ once all cuts with 2 edges have been considered.
- Conversely, if $G^\ddagger$ admits a 5-edge-colouring, then we consider, one at a time, every cut $F = \{uv, u'v'\}$ taken in an iteration of the reduction, transferring the colour of the edges uu' and vv' created by the reduction to the edges uv and $u'v'$, thus yielding the construction of a 5-edge-colouring of G. Clearly, this works if uu' and vv' are coloured the same, but if they are not, we recall that both sides of the

cut have been disconnected by the reduction, so we can permute the names of the colours in one of the sides in order to get the same colour on both uu' and vv'. □

Proof of Theorem 1. Since 5-snarks are by definition the negative instances of the problem EDGE-COLOURING(5-regular, 4-edge-connected), if they do not exist then this problem admits an $O(1)$-time algorithm, which by Theorem 3 implies $\mathcal{P} = \mathcal{NP}$. □

Corollary 2 follows from the classical Fortune–Mahaney's Theorem on the computational complexity of sparse languages [Fortune 1979, Mahaney 1982]. A language L is said to be *sparse* if there is a polynomial $p(n)$ such that the number of words of length n belonging to L is bounded above by $p(n)$ for any n. The Fortune–Mahaney's Theorem states that if any sparse language is $\mathcal{NP}$-complete or co$\mathcal{NP}$-complete, then $\mathcal{P} = \mathcal{NP}$.

Proof of Corollary 2. Follows by observing that the proof of Theorem 1 immediately implies that the language of the 5-snarks is co$\mathcal{NP}$-complete, so it cannot be sparse. □

We conclude observing the importance of 5-snarks in the context of the Overfull and the 1-Factorisation Conjectures, as briefly discussed in Section 1, and encouraging further investigation on these graphs and other interesting questions which arise: *Do 5-snarks contain 3-snarks as subgraphs? Do they allow infinite family constructions?*

Observation 4. *No 5-snark can be* SO.

Proof. A graph G on n vertices is *overfull* if and only if $\sum_{u\in V(G)}(\Delta(G) - d_G(u)) \leq \Delta(G) - 2$ and n is odd [Niessen 1994]. Ergo, if a 5-snark is *SO*, then it has an overfull subgraph H with $\Delta(H) = 5$ and, since 5-regular graphs have even order, $V(H) \neq V(G)$. Further assuming, without loss of generality, that H is induced by $V(H) =: U$, we have that $s := \sum_{u\in V(H)}(\Delta(H) - d_H(u))$ is the number of edges $uv \in E(G)$ with $u \in U$ and $v \notin U$, so $s \leq 3$ by the overfullness of H, but $s \geq 4$ by Definition 1, a contradiction. □

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References

Appel, K. and Haken, W. (1977). Every planar map is four colorable. Part I: discharging. *Illinois J. Math.*, 21:429–490.

Appel, K., Haken, W., and Koch, J. (1977). Every planar map is four colorable. Part II: reducibility. *Illinois J. Math.*, 21:491–567.

Arora, S. and Borak, B. (2007). *Computational Complexity: A Modern Approach*. Princeton.

Blanuša, D. (1946). Problem četiriju boja. *Glasnik. Mat. Fiz. Astr. Ser. II*, 1:31–42.

Chladný, M. and Škoviera, M. (2010). Factorisation of snarks. *Electron. J. Combin.*, 17(#R32).

Cook, S. A. (1971). The complexity of theorem-proving procedures. In *Proc. 3rd Annual ACM Symposium on Theory of Computing (STOC '71)*, pages 151–158, New York. ACM.

Descartes, B. (1948). Network-colourings. *The Mathematical Gazette*, 32:67–69.

Diestel, R. (2010). *Graph Theory*. Springer, 4^{th} edition.

Edmonds, J. and Karp, R. M. (1972). Theoretical improvements in algorithmic efficiency for network flow problems. *J. ACM*, 19:248–264.

Ford, L. R. and Fulkerson, D. R. (1956). Maximal flow through a network. *Can. J. Math.*, 8:399–404.

Fortune, S. (1979). A note on sparse complete sets. *SIAM J. Comput.*, 5:431–433.

Fu, Y. and Ma, Y. (2013). *Graph Embedding for Pattern Analysis*. Springer.

Gardner, M. (1976). Mathematical games. *Scientific American*, 4(234):126–130.

Holyer, I. (1981). The $\mathcal{NP}$-completeness of edge-colouring. *SIAM J. Comput.*, 10(4):718–720.

Isaacs, R. (1975). Infinite families of non-trivial trivalent graphs which are not Tait-colorable. *Amer. Math. Monthly*, 82:221–239.

Karp, R. M. (1972). Reducibility among combinatorial problems. In Miller, R. E. and Thatcher, J. W., editors, *Complexity of Computer Computations*, pages 85–103. Plenum.

Kempe, A. B. (1886). A memoir on the theory of mathematical form. *Philos. Trans. Royal Soc.*, 177:1–70.

Leven, D. and Galil, Z. (1983). $\mathcal{NP}$-completeness of finding the chromatic index of regular graphs. *J. Algorithms*, 4:35–44.

Lewis, R. M. R. (2016). *A Guide to Graph Colouring*. Springer.

Mahaney, S. R. (1982). Sparse complete sets of $\mathcal{NP}$: Solution of a conjecture by Berman and Hartmanis. *J. Comput. Syst. Sci.*, 25:130–143.

Niessen, T. (1994). How to find overfull subgraphs in graphs with large maximum degree. *Discrete Appl. Math.*, 51:117–125.

Petersen, J. (1898). Sur le théorème de Tait. *L'Intermédiaire des Mathématiciens*, 5:225–227.

Robinson, R. W. and Wormald, N. C. (1994). Almost all regular graphs are hamiltonian. *Random Struct. Algor.*, 5:363–374.

Skupień, Z. (2007). Exponentially many hypohamiltonian snarks. *Electron. Notes Discrete Math.*, 28:417–424.

Tait, P. G. (1878–1880). On the colouring of maps. *Proc. Roy. Soc. Edinburgh Sect. A*, 10:501–503,729.

Vizing, V. G. (1964). On an estimate of the chromatic class of a p-graph (in Russian). *Diskret. Analiz.*, 3:25–30.

Zatesko, L. M. (2018). *Novel Procedures for Graph Edge-colouring*. PhD thesis, Federal University of Paraná.